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# Differential calculi on the quantum group $GL_{p,q}(2)$

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**Abstract.** We present a systematical study of left-covariant differential calculus on the quantum group $GL_{p,q}(2)$, a two-parameter deformation of the general linear group. In particular, we explicitly construct the most general bicovariant calculus. It depends on a new parameter $s$. The corresponding 'Lie algebra' of left-coinvariant vector fields is a two-parameter deformation of the classical Lie algebra. $p$ and $q$ appear only in the combination $r = pq$. In the limit $p, q \to 1$ one obtains non-standard bicovariant differential calculi on the classical Lie group.

## 1. Introduction

The concept of a space(time) continuum has been a major ingredient in all successful physical theories. Nevertheless, there are some arguments that on a submicroscopic scale this concept has to be abandoned (see also [1]). Spacetime intervals are measured with (test) particles. But the resolution is limited by the quantum mechanical wavelike properties of particles. In order to probe spacetime at smaller and smaller length scales one needs heavier and heavier particles, corresponding to smaller and smaller wavelength. But as these become heavier they cannot be regarded as test particles any longer since their effect on the spacetime (curvature) is not negligible any more. The assumption that spacetime is smooth down to arbitrarily small distances is therefore without experimental support. On the other hand, this idealization might be responsible for the unsurmountable problems which one meets if one tries to understand the gravitational interaction as mediated by 'gravitons' considered to be similar as other particles.

This motivates us to look for a concept which could replace the notion of spacetime and on which a physical theory could be based. An indication how to do this originates from the following consideration (see also [2, 3]). If at a sufficiently small length scale coordinates become non-commuting operators, it will be impossible to measure the position of a particle exactly. In this way one may hope to overcome the ultraviolet divergencies of conventional quantum field theory which are due to the—in principle—possibility of measuring field oscillations at one point. We therefore expect non-commutativity to be an essential ingredient of the generalized spacetime concept. It is conceivable that such a non-commutativity at small length scales is due to a kind of 'quantization' independent of what we are used to understand as quantization. But it may well turn out that there is a deeper relation between these two quantizations.

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A general framework in which such an approach can be pursued is non-commutative geometry [4–6]†. The basic observation is that a manifold is completely characterized by the (commutative) algebra of functions on it. Geometrical concepts like exterior and covariant derivative can then be formulated as operations on this algebra and generalized to non-commutative algebras (for which there is no longer an underlying manifold).

The best-understood examples are groups which, reformulated in this algebraic sense, are turned into commutative Hopf algebras [12]. 'Quantum groups' [13–16] are (special kinds of) non-commutative Hopf algebras which are deformations of classical groups (as Hopf algebras). These are not mathematical artifacts. They appear in the quantum inverse scattering method (see [17] for an introduction), in two-dimensional conformal field theories and certain integrable models [18]. Quantum groups generalize our more familiar concepts of symmetries to the realm of non-commutative geometry [14]. Presuming a kind of non-commutative version of Einstein's general relativity, we expect quantum groups to take over the role of isometry and gauge groups. Certain quantum groups may even serve as generalized spacetime models.

A major step in such a programme is to develop differential calculus on quantum groups. This has been done by Woronowicz [19, 20] and described in an easier accessible way by Wess and Zumino [21–23] (see also [24–27]). Whereas there is a distinguished classical differential calculus, this uniqueness is lost if non-commutativity is introduced. There are many different differential calculi on a quantum group. On a classical Lie group there is a left and a right action of the group on itself and these commute. The requirement of such a 'bicovariance' [20] on a quantum group restricts the possible differential calculi. However, as we will show for the example of the two-parameter deformation of GL(2) [23, 28, 29] this condition does not fix the differential calculus. There are two 1-parameter families of bicovariant differential calculi.

In section 2 we briefly recall how groups are reformulated as Hopf algebras, concentrating on the example of the (complex) general linear group GL(2). Section 3 introduces its two-parameter deformation $\mathrm{GL}_{p,q}(2)$ and some formulae needed in the following. These two sections do not contain new results.

Differential calculus on quantum groups is the subject of section 4. Instead of starting with the general, rather abstract formalism [20] we develop the calculus from the example $\mathrm{GL}_{p,q}(2)$ in a more pedagogical way, systematically extending recent work by Schirrmacher, Wess and Zumino [23]. They presented an example of a right-covariant differential calculus. A somewhat simpler example (of a left-covariant calculus) is presented in section 5 and a deformation (as a Hopf algebra) of the Lie algebra of GL(2) is derived as the algebra of left-coinvariant vector fields on the quantum group. The corresponding analysis parallels that in [23].

In section 6 we construct all first-order bicovariant differential calculi [20] on $\mathrm{GL}_{p,q}(2)$. A special example of a bicovariant calculus on this quantum group has been communicated previously [29, 30]. Section 7 recalls some results on higher-order differential calculus from the work of Woronowicz [20] and thereby paves the way for the calculation of the 1-form commutation relations (and thus the generalized wedge product) for the general bicovariant calculus on $\mathrm{GL}_{p,q}(2)$. This is done in section 8.

Some examples are extracted in section 9 from these general results. They include non-standard bicovariant differential calculi on the classical Lie group GL(2). Finally, some conclusions and further remarks are collected in section 10.

† For some examples of applications of non-commutative geometry we also refer to [2, 7–11].

## 2. Groups as Hopf algebras

Since quantum groups are considered as (special examples of) non-commutative Hopf algebras, one first has to understand in which sense ordinary (Lie) groups can be viewed as commutative Hopf algebras. The general idea behind it is a reformulation of structures of a manifold $M$ in terms of the algebra $\mathscr{A}$ of functions on $M$†. Let us consider the group $\mathrm{G}=\mathrm{GL}(2,\mathbb{C})$. An element is represented by an invertible matrix

$$g:=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \tag{2.1}$$

with complex entries. If $f$ is a ($\mathbb{C}$-valued) function on G, the group multiplication $m$: $\mathrm{G}\times\mathrm{G}\to\mathrm{G}$ can be reformulated as an operation $\Delta$: $\mathscr{A}\to\mathscr{A}\otimes\mathscr{A}$ such that

$$\Delta(f)(g,g'):=f(m(g,g')) \qquad \forall g,g'\in\mathrm{G} \tag{2.2}$$

respectively

$$\Delta(f)=f\circ m. \tag{2.3}$$

$\Delta$ is an algebra homomorphism since

$$\Delta(f+h)=\Delta(f)+\Delta(h) \qquad \Delta(fh)=\Delta(f)\Delta(h) \qquad \Delta(cf)=c\Delta(f) \tag{2.4}$$

for any two functions $f$, $h$ on G and $c\in\mathbb{C}$. Because of this property the 'co-product' $\Delta$ is already fixed if we know its action on a basic set of functions. In our special case $\mathrm{G}=\mathrm{GL}(2,\mathbb{C})$ we have a natural choice, namely

$$\begin{aligned} a\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)&:=\alpha \qquad & b\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)&:=\beta\\ c\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)&:=\gamma \qquad & d\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)&:=\delta. \end{aligned} \tag{2.5}$$

As functions on G we consider *polynomials*‡ of the 'elementary functions' $a$, $b$, $c$, $d$ and the 'identity function' $\mathbb{1}$. A linear operator on the algebra $\mathscr{A}$ is then determined by its action on the monomials $a^k d^l b^m c^n$ (where $k$, $l$, $m$, $n\in\mathbb{N}\cup\{0\}$ and $a^0=\mathbb{1}$ etc).

In order to calculate the co-product of $a$, $b$, $c$, $d$, we have to consider

$$\Delta\begin{pmatrix}a & b\\ c & d\end{pmatrix}\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}\right)=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\cdot\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}\right) \tag{2.6}$$

so that

$$\Delta\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a\otimes a+b\otimes c & a\otimes b+b\otimes d\\ c\otimes a+d\otimes c & c\otimes b+d\otimes d\end{pmatrix}. \tag{2.7}$$

Here we have used a compact notation for

$$\Delta(a)=a\otimes a+b\otimes c \tag{2.8}$$

etc. In addition, we have

$$\Delta(\mathbb{1})=\mathbb{1}\otimes\mathbb{1}. \tag{2.9}$$

† This is an algebra over $\mathbb{C}$ with the pointwise addition and multiplication of functions.

‡ We will not discuss the extension to larger classes of functions. This is a delicate point for non-compact quantum groups.

In a similar way as the group multiplication is translated into a homomorphism of the algebra $\mathscr{A}$ (the co-product) the existence of a unit group element is reformulated into the 'co-unit' algebra homomorphism

$$\varepsilon: \mathscr{A} \to \mathbb{C} \qquad \varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \varepsilon(\mathbb{1}) = 1. \tag{2.10}$$

Furthermore, the existence of an inverse group element translates to an algebra (anti-) homomorphism†

$$S: \mathscr{A} \to \mathscr{A} \qquad S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)^{-1}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad S(\mathbb{1}) = \mathbb{1} \tag{2.11}$$

which is called 'antipode'. Again, equations (2.10) and (2.11) are compact notation for

$$\varepsilon(a) = 1 \qquad \varepsilon(b) = \varepsilon(c) = 0 \qquad \varepsilon(d) = 1 \tag{2.12}$$

$$S(a) = d/(ad - bc) \text{ etc.} \tag{2.13}$$

The co-product, co-unit and antipode supply $\mathscr{A}$ with a Hopf algebra structure [12].

*Definition.* A $\mathbb{C}$-algebra $\mathscr{A}$ is a Hopf algebra, if there is

(1) a $\mathbb{C}$-algebra homomorphism (co-product) $\Delta: \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ which is 'co-associative'‡:

$$(\Delta \otimes \mathbb{1}) \circ \Delta = (\mathbb{1} \otimes \Delta) \circ \Delta \tag{2.14}$$

(2) a $\mathbb{C}$-algebra homomorphism (co-unit) $\varepsilon: \mathscr{A} \to \mathbb{C}$ such that

$$(\mathbb{1} \otimes \varepsilon) \circ \Delta = \mathbb{1} \qquad (\varepsilon \otimes \mathbb{1}) \circ \Delta = \mathbb{1} \tag{2.15}$$

(note that $\mathscr{A} \otimes \mathbb{C} = \mathbb{C} \otimes \mathscr{A} = \mathscr{A}$),

(3) a $\mathbb{C}$-algebra antihomomorphism (antipode) $S: \mathscr{A} \to \mathscr{A}$ which satisfies

$$\mu[(\mathbb{1} \otimes S) \circ \Delta] = \varepsilon \qquad \mu[(S \otimes \mathbb{1}) \circ \Delta] = \varepsilon \tag{2.16}$$

where $\mu$ stands for the algebra product $\mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$.

We will also demand that (2.9) holds. Then $\varepsilon(\mathbb{1}) = 1$ and $S(\mathbb{1}) = \mathbb{1}$. The Hopf algebra structure as defined above only requires that $\Delta(\mathbb{1})$ is idempotent (i.e. $\Delta(\mathbb{1})^2 = \Delta(\mathbb{1})$) and commutes with all co-products. Weaker Hopf algebras in this sense have been considered in [31].

## 3. The quantum group $GL_{p,q}(2)$

'Coordinates' $x$ and $y$ satisfying

$$xy = qyx \tag{3.1}$$

with a complex parameter $q$ are said to span a 'quantum plane' [14, 21]. Consider a transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{3.2}$$

† Here the algebra $\mathscr{A}$ is commutative, so this is a homomorphism as well as an antihomomorphism.

‡ Here and in the following we use $\mathbb{1}$ (which was introduced as the unit in $\mathscr{A}$) to also denote the identity map on $\mathscr{A}$.

which preserves the relation (3.1). We assume that $a$, $b$, $c$, $d$ commute† with the coordinates $x$ and $y$. This leads to

$$ac = qca \qquad bd = qdb \qquad ad = da + qcb - (1/q)bc \tag{3.3}$$

which does not determine all commutation relations between the entries of the transformation matrix. Let us therefore extend the quantum plane structure by introducing 'differentials'

$$\xi := \mathrm{d}x \qquad \eta := \mathrm{d}y \tag{3.4}$$

such that

$$\xi\eta + (1/p)\eta\xi = 0 \qquad \xi^2 = \eta^2 = 0 \tag{3.5}$$

where $p$ is another parameter‡. In the limit $p \to 1$ we then recover the familiar anticommutation of differentials. Assuming that the exterior derivative d commutes with $a$, $b$, $c$, $d$ and requiring that the transformed differentials also satisfy the relations (3.5), we find

$$bc = (q/p)cb \qquad ab = pba \qquad cd = pdc. \tag{3.6}$$

Equations (3.3) and (3.6) are the commutation relations of a two-parameter deformation of GL(2, $\mathbb{C}$) [23, 28, 29]. Co-product and co-unit are defined, respectively, by§

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\otimes a + b\otimes c & a\otimes b + b\otimes d \\ c\otimes a + d\otimes c & c\otimes b + d\otimes d \end{pmatrix} \tag{3.7}$$

$$\varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.8}$$

in analogy with the commutative case described in section 2.

The relation (2.16) for an antipode demands

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix} &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \\ \begin{pmatrix} S(a) & S(b) \\ S(c) & S(d) \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \end{aligned} \tag{3.9}$$

which means that the matrix with entries $S(a)$, $S(b)$, $S(c)$, $S(d)$ has to be a left and right inverse for the matrix with entries $a$, $b$, $c$, $d$. In the 'classical' case the inverse is constructed using the determinant of the latter matrix. A generalized determinant

$$\mathcal{D} := \det_q\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.10}$$

can be defined in the 'quantum' case by the transformation formula

$$\xi'\eta' = \mathcal{D}\xi\eta \tag{3.11}$$

so that

$$\mathcal{D} = ad - pbc = da - q^{-1}bc. \tag{3.12}$$

† This is automatically implemented if we understand the products on the RHS of (3.2) as tensor products, e.g. $x' = a\otimes x + b\otimes y$.

‡ More general relations than (3.1) and (3.5) may be considered.

§ This has to be supplemented by the 'trivial' definitions $\Delta(\mathbb{1}) = \mathbb{1}\otimes\mathbb{1}$ and $\varepsilon(\mathbb{1}) = 1$.

The commutation relations with $a$, $b$, $c$, $d$ are given by

$$\mathcal{D}a = a\mathcal{D} \qquad \mathcal{D}d = d\mathcal{D} \qquad \mathcal{D}b = (p/q)b\mathcal{D} \qquad \mathcal{D}c = (q/p)c\mathcal{D}. \tag{3.13}$$

In order to construct an antipode, we have to assume that $\mathcal{D}$ has an inverse $\mathcal{D}^{-1}$†:

$$\mathcal{D}^{-1}\mathcal{D} = \mathbb{1} = \mathcal{D}\mathcal{D}^{-1}. \tag{3.14}$$

Acting on the commutation relations (3.13) with $\mathcal{D}^{-1}$ from left and right, we find

$$\begin{aligned} &a\mathcal{D}^{-1} = \mathcal{D}^{-1}a \qquad d\mathcal{D}^{-1} = \mathcal{D}^{-1}d \\ &b\mathcal{D}^{-1} = (p/q)\mathcal{D}^{-1}b \qquad c\mathcal{D}^{-1} = (q/p)\mathcal{D}^{-1}c \end{aligned} \tag{3.15}$$

and from

$$\Delta(\mathcal{D}) = \mathcal{D}\otimes\mathcal{D} \qquad \Delta(\mathcal{D})\Delta(\mathcal{D}^{-1}) = \Delta(\mathbb{1}) = \mathbb{1}\otimes\mathbb{1} \tag{3.16}$$

we obtain

$$\Delta(\mathcal{D}^{-1}) = \mathcal{D}^{-1}\otimes\mathcal{D}^{-1}. \tag{3.17}$$

Now the antipode is determined by

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathcal{D}^{-1}\begin{pmatrix} d & -b/q \\ -qc & a \end{pmatrix} = \begin{pmatrix} d & -b/p \\ -pc & a \end{pmatrix}\mathcal{D}^{-1} \tag{3.18}$$

and we have

$$S(\mathcal{D}) = \mathcal{D}^{-1} \qquad S(\mathcal{D}^{-1}) = \mathcal{D}. \tag{3.19}$$

Although the antipode has some properties of an inverse, we have $S^2 \neq \mathbb{1}$ (see also [13]). Indeed,

$$\begin{aligned} &S^{-1}(a) = S(a) \qquad S^{-1}(b) = pqS(b) \\ &S^{-1}(c) = \frac{1}{pq}S(c) \qquad S^{-1}(d) = S(d). \end{aligned} \tag{3.20}$$

One still has to check consistency in the following sense. Are all the monomials $a^k d^l b^m c^n$ functionally independent?‡ The problem is that there are several different ways of reordering a given monomial having more than two different generators by using the commutation relations. If these do not lead to identical results, we obtain non-trivial relations between the generators or restrictions on the deformation parameters. For example, the two possibilities

$$\begin{array}{ccccc} & cab & \rightarrow & acb & \\ \nearrow & & & & \searrow \\ cba & & & & abc \\ \searrow & & & & \nearrow \\ & bca & \rightarrow & bac & \end{array}$$

of reordering $cba$ into $abc$ indeed lead to the same result. According to Manin [32] it is actually sufficient to check consistency for the *cubic* monomials (see also [28]). This is easily done in the case under consideration. The resulting Hopf algebra will be denoted as $GL_{p,q}(2)$.

† More precisely, we extend the algebra by an element $\mathcal{D}^{-1}$ satisfying the relations listed in the following.
‡ In the case of Lie algebras the corresponding result is known as the Poincaré-Birkhoff-Witt theorem.

The commutation relations of $\mathrm{GL}_{p,q}(2)$ can be expressed with the help of the $\hat{R}$-matrix

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-p^{-1} & qp^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{3.21}$$

as follows

$$\hat{R}^{\alpha\beta}{}_{\gamma\delta} M^{\gamma}{}_{\mu} M^{\delta}{}_{\nu} = M^{\alpha}{}_{\gamma} M^{\beta}{}_{\delta} \hat{R}^{\gamma\delta}{}_{\mu\nu} \qquad (\alpha, \beta \ldots = 1, 2) \tag{3.22}$$

with

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.23}$$

The entries of the matrix $\hat{R}$ are numbered as (1, 1), (1, 2), (2, 1), (2, 2). For example, $\hat{R}^{12}{}_{21} = qp^{-1}$. The matrix $\hat{R}$ is a solution of the quantum Yang–Baxter equation [23, 28].

## 4. Differential calculus on $\mathrm{GL}_{p,q}(2)$

The central object of differential calculus is the exterior derivative

$$\mathrm{d}: \qquad \mathscr{A} \to \Lambda^1(\mathscr{A}) = \text{space of 1-forms} \tag{4.1}$$

satisfying $\mathrm{d}^2 = 0$ and the Leibniz rule

$$\mathrm{d}(fh) = (\mathrm{d}f)h + f\,\mathrm{d}h \qquad \forall f, h \in \mathscr{A}. \tag{4.2}$$

In order for this equation to make sense,we have to be able to multiply 1-forms from the left and from the right by elements of $\mathscr{A}$. Therefore $\Lambda^1(\mathscr{A})$ must be an $\mathscr{A}$-bimodule.

In differential geometry of the classical Lie groups left (and right) invariant (Maurer–Cartan) 1-forms play a special role. To find their quantum analogues we have to translate first the group left action on 1-forms to a left-coaction:

$$\Delta_{\mathscr{L}}: \qquad \Lambda^1(\mathscr{A}) \to \mathscr{A} \otimes \Lambda^1(\mathscr{A}). \tag{4.3}$$

This should be a bimodule homomorphism, i.e.

$$\begin{aligned} \Delta_{\mathscr{L}}(f\omega) &= \Delta(f)\Delta_{\mathscr{L}}(\omega) \\ \Delta_{\mathscr{L}}(\omega f) &= \Delta_{\mathscr{L}}(\omega)\Delta(f) \end{aligned} \qquad \forall f \in \mathscr{A}, \omega \in \Lambda^1(\mathscr{A}) \tag{4.4}$$

such that

$$\Delta_{\mathscr{L}}\,\mathrm{d} = (\mathbb{1} \otimes \mathrm{d})\Delta. \tag{4.5}$$

To understand the last condition, let us look at the case $\mathrm{GL}_{p,q}(2)$:

$$\begin{aligned} \Delta_{\mathscr{L}} \begin{pmatrix} \mathrm{d}a & \mathrm{d}b \\ \mathrm{d}c & \mathrm{d}d \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} \mathrm{d}a & \mathrm{d}b \\ \mathrm{d}c & \mathrm{d}d \end{pmatrix} \\ &= \begin{pmatrix} a \otimes \mathrm{d}a + b \otimes \mathrm{d}c & a \otimes \mathrm{d}b + b \otimes \mathrm{d}d \\ c \otimes \mathrm{d}a + d \otimes \mathrm{d}c & c \otimes \mathrm{d}b + d \otimes \mathrm{d}d \end{pmatrix}. \end{aligned} \tag{4.6}$$

This is what we should expect as the left coaction on the differentials (which generate $\Lambda^1(\mathscr{A})$). A 1-form $\theta$ is 'left-coinvariant' if

$$\Delta_{\mathscr{L}}\theta = \mathbb{1}\otimes\theta. \tag{4.7}$$

In analogy to the 'classical' case we introduce left-coinvariant Maurer–Cartan forms on $\mathrm{GL}_{p,q}(2)$ by

$$\begin{aligned}\begin{pmatrix}\theta^1 & \theta^2\\ \theta^3 & \theta^4\end{pmatrix} &= S\begin{pmatrix}a & b\\ c & d\end{pmatrix}\mathrm{d}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\\ &= \begin{pmatrix}S(a)\,\mathrm{d}a+S(b)\,\mathrm{d}c & S(a)\,\mathrm{d}b+S(b)\,\mathrm{d}d\\ S(c)\,\mathrm{d}a+S(d)\,\mathrm{d}c & S(c)\,\mathrm{d}b+S(d)\,\mathrm{d}d\end{pmatrix}\\ &= \mathscr{D}^{-1}\begin{pmatrix}d\,\mathrm{d}a-q^{-1}b\,\mathrm{d}c & d\,\mathrm{d}b-q^{-1}b\,\mathrm{d}d\\ -qc\,\mathrm{d}a+a\,\mathrm{d}c & -qc\,\mathrm{d}b+a\,\mathrm{d}d\end{pmatrix}.\end{aligned} \tag{4.8}$$

Let us check that these 1-forms are indeed invariant under the left-coaction†

$$\begin{aligned}\Delta_{\mathscr{L}}\theta^1 &= \Delta_{\mathscr{L}}(S(a)\,\mathrm{d}a+S(b)\,\mathrm{d}c)\\ &= \Delta\circ S(a)\Delta_{\mathscr{L}}(\mathrm{d}a)+\Delta\circ S(b)\Delta_{\mathscr{L}}(\mathrm{d}c)\\ &= \Delta(\mathscr{D}^{-1})[\Delta(d)(\mathbb{1}\otimes\mathrm{d})\Delta(a)-q^{-1}\Delta(b)(\mathbb{1}\otimes\mathrm{d})\Delta(c)]\\ &= \Delta(\mathscr{D}^{-1})[(cb-q^{-1}ad)\otimes b\,\mathrm{d}c+(da-q^{-1}bc)\otimes d\,\mathrm{d}a]\\ &= \Delta(\mathscr{D}^{-1})\mathscr{D}\otimes\mathscr{D}\theta^1\\ &= \mathbb{1}\otimes\theta^1.\end{aligned} \tag{4.9}$$

Similarly we obtain

$$\Delta_{\mathscr{L}}\theta^K = \mathbb{1}\otimes\theta^K \qquad (K=1,\ldots,4). \tag{4.10}$$

Applying d to the commutation relations of $\mathrm{GL}_{p,q}(2)$ leads to

$$(\mathrm{d}a)b+a(\mathrm{d}b) = p(\mathrm{d}b)a+pb\,\mathrm{d}a \tag{4.11}$$

etc. Using (3.9) the defining formula for the $\theta^K$ can be inverted to

$$\mathrm{d}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\theta^1 & \theta^2\\ \theta^3 & \theta^4\end{pmatrix}. \tag{4.12}$$

*Remark.* The differentials of $a$, $b$, $c$, $d$ satisfy the general formula [20]

$$\mathrm{d}f = (\chi_K * f)\theta^K = (\mathbb{1}\otimes\chi_K)\Delta(f)\theta^K \qquad (f\in\mathscr{A}) \tag{4.13}$$

(summation over $K$). We have

$$\chi_K(a_L) = \delta_{KL} \tag{4.14}$$

with $a_1=a$, $a_2=b$, $a_3=c$, $a_4=d$. □

† This is the only calculation where we use (4.5).

Equation (4.12) allows us to express the differentiated commutation relations in terms of the $\theta^K$ as follows

$$a\theta^1 b - ab\theta^1 + a^2\theta^2 - pa\theta^2 a - pb^2\theta^3 + b\theta^3 b + ab\theta^4 - pb\theta^4 a = 0$$

$$a\theta^1 c - qc\theta^1 a + (ad - pbc)\theta^3 + b\theta^3 c - qd\theta^3 a = 0$$

$$\begin{aligned} a\theta^1 d - ad\theta^1 + ac\theta^2 - c\theta^2 a + (q^{-1} - p)a\theta^2 c - pbd\theta^3 + b\theta^3 d \\ + ad\theta^4 - d\theta^4 a + (q^{-1} - p)b\theta^4 c = 0 \end{aligned} \tag{4.15}$$

$$qc\theta^1 b - pbc\theta^1 + ac\theta^2 - pa\theta^2 c - pbd\theta^3 + qd\theta^3 b + pbc\theta^4 - pb\theta^4 c = 0$$

$$q(ad - pbc)\theta^2 - a\theta^2 d + qc\theta^2 b + qd\theta^4 b - b\theta^4 d = 0$$

$$c\theta^1 d - cd\theta^1 + c^2\theta^2 - pc\theta^2 c - pd^2\theta^3 + d\theta^3 d + cd\theta^4 - pd\theta^4 c = 0.$$

These equations restrict the commutation relations between left-coinvariant 1-forms and elements of $\mathscr{A}$. But they do not fix the commutation structure completely. Let us specify a commutation structure by†

$$\theta^K f = \Theta(f)^K_L \theta^L \qquad (\forall f \in \mathscr{A}) \tag{4.16}$$

where $\Theta(f)$ is a matrix with entries in $\mathscr{A}$, respectively $\Theta$ a map from $\mathscr{A}$ to $\mathscr{A}$. From

$$\Theta(fh)^K_L \theta^L = \theta^K fh = (\theta^K f)h = \Theta(f)^K_L \theta^L h = \Theta(f)^K_M \Theta(h)^M_L \theta^L \tag{4.17}$$

we conclude that

$$\Theta(fh) = \Theta(f)\Theta(h) \qquad \forall f, h \in \mathscr{A} \tag{4.18}$$

in matrix notation. Furthermore, (4.4) implies

$$\begin{aligned} (\mathbb{1} \otimes \theta^K)\Delta(f) &= \Delta_{\mathscr{L}}(\theta^K)\Delta(f) = \Delta_{\mathscr{L}}(\theta^K f) \\ &= \Delta_{\mathscr{L}}(\Theta(f)^K_L \theta^L) = \Delta(\Theta(f)^K_L)(\mathbb{1} \otimes \theta^L). \end{aligned} \tag{4.19}$$

Using the general form of the co-product

$$\Delta(f) = \sum_m f_m \otimes f'_m \tag{4.20}$$

we obtain

$$\begin{aligned} (\mathbb{1} \otimes \theta^K)\Delta(f) &= \sum_m f_m \otimes \theta^K f'_m \\ &= \sum_m f_m \otimes \Theta(f'_m)^K_L \theta^L \\ &= \left(\sum_m f_m \otimes \Theta(f'_m)^K_L\right)(\mathbb{1} \otimes \theta^L) \\ &= (\mathbb{1} \otimes \Theta^K_L)\Delta(f)(\mathbb{1} \otimes \theta^L) \end{aligned} \tag{4.21}$$

so that [20, 24]

$$\Delta \circ \Theta = (\mathbb{1} \otimes \Theta)\Delta. \tag{4.22}$$

† Underlying is the assumption that the $\theta^K$ form a basis of the space of 1-forms. In the following we use the summation convention. For example, summation over $L$ is understood in (4.16).

*Let us investigate these constraints on the matrix $\Theta$ (which determines the commutation* structure between 1-forms and elements of the algebra $\mathscr{A}$) in the case of $GL_{p,q}(2)$. First we note that

$$\Theta(\mathbb{1})^K_L = \delta^K_L \mathbb{1}. \tag{4.23}$$

Inserting the general expression

$$\Theta(a)^K_L = \sum A^K_{L,k,l,m,n} a^k d^l b^m c^n \tag{4.24}$$

in (4.22) leads to

$$\Theta(a) = aA + bC \qquad \Theta(c) = cA + dC \tag{4.25}$$

with $4\times 4$-matrices $A$ and $C$ (with entries in $\mathbb{C}$). Similarly, we find

$$\Theta(b) = aB + bD \qquad \Theta(d) = cB + dD \tag{4.26}$$

with matrices $B$ and $D$. This is all we get from applying (4.22) to $a$, $b$, $c$ and $d$†. The result can be written in the compact form

$$\Theta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{4.27}$$

Acting with $\Theta$ on the commutation relations of the algebra (or using (4.18)), one finds that $A$, $B$, $C$, $D$ satisfy the same commutation relations as $a$, $b$, $c$, $d$. In other words, $A$, $B$, $C$, $D$ must be a representation of $GL_{p,q}(2)$ with ordinary (complex) $4\times 4$-matrices.

*Remark.* If we denote this representation by $\mathscr{F}$, then (4.27) can be written as a convolution product:

$$\Theta = \mathscr{F}\star = (\mathbb{1}\otimes\mathscr{F})\Delta \tag{4.28}$$

in accordance with a general result due to Woronowicz [20]. Then

$$\begin{pmatrix} \varepsilon & \chi_K \\ 0 & \mathscr{F}^K_L \end{pmatrix} \tag{4.29}$$

(with the counit $\varepsilon$ and $\chi_K$ defined in (4.14)) is also a representation of $\mathscr{A}$ [19, 24]. □

Later we will also need the action of $\Theta$ on the quantum determinant, i.e.

$$\Theta(\mathscr{D}) = \mathscr{D}(AD - pBC) \qquad \Theta(\mathscr{D}^{-1}) = \mathscr{D}^{-1}(AD - pBC)^{-1}. \tag{4.30}$$

Now we can use the above results to commute all the Maurer–Cartan forms in (4.15) to the right and obtain the following restrictions on the entries of the matrices $A$, $B$, $C$, $D$:

$$\begin{aligned}
&A^2{}_1 = p^{-1}B^1{}_1 && B^3{}_1 = pq^{-1}C^2{}_1 \\
&A^2{}_2 = p^{-1}(B^1{}_2 + 1) && B^3{}_2 = pq^{-1}C^2{}_2 \\
&A^2{}_3 = p^{-1}B^1{}_3 && B^3{}_3 = pq^{-1}C^2{}_3 \\
&A^2{}_4 = p^{-1}B^1{}_4 && B^3{}_4 = pq^{-1}C^2{}_4 \\
&A^3{}_1 = q^{-1}C^1{}_1 && D^2{}_1 = qB^4{}_1
\end{aligned}$$

† The homomorphism properties of $\Delta$ and $\Theta$ then guarantee that (4.22) ((4.19) respectively) holds for any $f\in\mathscr{A}$.

$$
\begin{array}{ll}
A^3{}_2 = q^{-1}C^1{}_2 & D^2{}_2 = q(B^4{}_2+1) \\
A^3{}_3 = q^{-1}(C^1{}_3+1) & D^2{}_3 = qB^4{}_3 \\
A^3{}_4 = q^{-1}C^1{}_4 & D^2{}_4 = qB^4{}_4 \\
A^4{}_1 = (q^{-1}-p)C^2{}_1+D^1{}_1-1 & D^3{}_1 = pC^4{}_1 \\
A^4{}_2 = (q^{-1}-p)C^2{}_2+D^1{}_2 & D^3{}_2 = pC^4{}_2 \\
A^4{}_3 = (q^{-1}-p)C^2{}_3+D^1{}_3 & D^3{}_3 = p(C^4{}_3+1) \\
A^4{}_4 = (q^{-1}-p)C^2{}_4+D^1{}_4+1 & D^3{}_4 = pC^4{}_4.
\end{array}
\tag{4.31}
$$

In order to construct a (first-order) differential calculus we have to find a representation of $GL_{p,q}(2)$ consisting of $4\times 4$-matrices $A$, $B$, $C$, $D$ satisfying the relations listed above†. $A$ and $D$ should be invertible since otherwise there is no 'classical limit'. The commutation relations for $A$, $B$, $C$, $D$ (with $q\neq 1$ or $p\neq 1$) then imply $\det B = \det C = 0$.

In principle it is now possible to solve the commutation relations for $A$, $B$, $C$, $D$ to construct *all* left-covariant differential calculi. This is straightforward, but one has to distinguish many different cases and runs into a tedious analysis. Anyway, this results in a lot of different calculi, a simple example is presented in section 5. Some additional restrictions arise from the extension to higher-order forms as will be explained below. It is of more interest to find natural conditions to narrow down these possibilities. One such condition is 'bicovariance' which we exploit in detail in section 6.

So far we have not discussed whether all the differential calculi obtained in this way are really different. Indeed, different matrices $A$, $B$, $C$, $D$ (subject to the restrictions above) determine different differential calculi. This is seen as follows. Two differential calculi $(\Lambda^1, d)$ and $(\tilde{\Lambda}^1, \tilde{d})$ are said to be identical [20] if there is a bimodule isomorphism $\zeta:\Lambda^1\to\tilde{\Lambda}^1$ such that $\zeta(\mathrm{d}f)=\tilde{\mathrm{d}}f\,\forall f\in\mathcal{A}$. Acting with $\zeta$ on (4.12) leads to $\zeta(\theta^K)=\tilde{\theta}^K$ and applying $\zeta$ to (4.16) then shows that the commutation structure between left-coinvariant Maurer–Cartan 1-forms and elements of $\mathcal{A}$ must be the same for both calculi.

Acting with d on (4.12) and using $\mathrm{d}^2=0$ we obtain the Maurer–Cartan equation (in matrix notation)

$$
\mathrm{d}\theta = -\theta\theta \tag{4.32}
$$

respectively

$$
\mathrm{d}\theta^M = -C^M_{KL}\theta^K\theta^L \tag{4.33}
$$

with non-vanishing coefficients

$$
C^1_{11}=C^1_{23}=C^2_{12}=C^2_{24}=C^3_{31}=C^3_{43}=C^4_{32}=C^4_{44}=1. \tag{4.34}
$$

We have not yet specified the product between 1-forms. It is constrained by the equations which we obtain by applying d to (4.16) using the Maurer–Cartan equation and (4.12). Assuming that d anticommutes with 1-forms, so that

$$
\mathrm{d}(\omega f) = (\mathrm{d}\omega)f - \omega\,\mathrm{d}f \tag{4.35}
$$

† We then have a left-covariant bimodule, see the paragraph following equation (2.30) in [20].

we obtain the following equations:

$$
\begin{aligned}
&[A^K_L C^L_{MN} - C^K_{IJ}(A^I_M A^J_N + B^I_M C^J_N)]\theta^M\theta^N \\
&\qquad = A^K_L(\theta^1\theta^L + \theta^L\theta^1) + B^K_L\theta^L\theta^3 + C^K_L\theta^2\theta^L \\
&[C^K_L C^L_{MN} - C^K_{IJ}(C^I_M A^J_N + D^I_M C^J_N)]\theta^M\theta^N \\
&\qquad = A^K_L\theta^3\theta^L + C^K_L(\theta^4\theta^L + \theta^L\theta^1) + D^K_L\theta^L\theta^3 \\
&[B^K_L C^L_{MN} - C^K_{IJ}(A^I_M B^J_N + B^I_M D^J_N)]\theta^M\theta^N \\
&\qquad = A^K_L\theta^L\theta^2 + B^K_L(\theta^1\theta^L + \theta^L\theta^4) + D^K_L\theta^2\theta^L \\
&[D^K_L C^L_{MN} - C^K_{IJ}(C^I_M B^J_N + D^I_M D^J_N)]\theta^M\theta^N \\
&\qquad = B^K_L\theta^3\theta^L + C^K_L\theta^L\theta^2 + D^K_L(\theta^4\theta^L + \theta^L\theta^4).
\end{aligned}
\tag{4.36}
$$

Whereas the square of a 'classical' 1-form always vanishes, this need not be so for the 'quantum' 1-forms. In order to mimic the classical structure as close as possible it is perhaps a reasonable further assumption that the quantum version of the space of 2-forms should have the same dimension as in the classical case and that $\theta^I\theta^J$ with $I<J$ form a basis. Then, for example, $(\theta^1)^2$ can be expressed as a linear combination of these basis elements and should vanish in the classical limit ($p, q \to 1$). We will *not* use this additional assumption in the following. All our examples automatically turned out to be in accordance with it.

It is not sufficient to find commutation relations for the 1-forms $\theta^K$ such that (4.36) is satisfied. In addition one has to check consistency with the algebra $\mathcal{A}$. Commuting $a$, $b$, $c$, $d$ through the commutation relations for $\theta^K$ in general leads to additional restrictions (see the example in section 5).

'Quantum' analogues of left-invariant vector fields are defined by [19]

$$
\mathrm{d}f = (\nabla_K f)\theta^K \qquad (\forall f \in \mathcal{A}). \tag{4.37}
$$

The Leibniz rule for d leads to a graded Leibniz rule for the operators $\nabla_K : \mathcal{A} \to \mathcal{A}$,

$$
\nabla_K(fh) = (\nabla_L f)\Theta(h)^L_K + f\nabla_K h \tag{4.38}
$$

and (4.5) translates (using (4.20)) into the left-coinvariance condition

$$
\Delta \circ \nabla_K = (\mathbb{1} \otimes \nabla_K)\Delta. \tag{4.39}
$$

*Acting with* d *on* (4.37) *leads to*

$$
0 = (\nabla_K\nabla_L f)\theta^K\theta^L + (\nabla_M f)\,\mathrm{d}\theta^M = (\nabla_K\nabla_L - C^M_{KL}\nabla_M)f\theta^K\theta^L. \tag{4.40}
$$

Once we know the commutation relations for the 1-forms $\theta^K$ we can read off from the last equation the commutation relations for the $\nabla_K$.

In order to calculate the commutation relations with elements of $\mathcal{A}$ (cf [22, 23]), we interpret $\nabla_K$ as operators acting to the left, i.e. instead of $\nabla_K f$ we write $f\overleftarrow{\nabla}_K$. This is also suggested by the form of (4.38). A product of operators like $\nabla_K\nabla_L$ has then to be replaced by $\overleftarrow{\nabla}_L\overleftarrow{\nabla}_K$ and the Leibniz rule leads to

$$
(\overleftarrow{\nabla}_L\Theta(h)^L_K - h\overleftarrow{\nabla}_K)\theta^K + \mathrm{d}h = 0 \tag{4.41}
$$

where $h$ (and $\mathrm{d}h$) is understood as a multiplication operator. Evaluating this equation with $h = a, b, c, d$ and using (4.12) yields

$$
\begin{aligned}
a\overleftarrow{\nabla}_K &= \overleftarrow{\nabla}_L(A^L_K a + C^L_K b) + \delta^1_K a + \delta^3_K b \\
b\overleftarrow{\nabla}_K &= \overleftarrow{\nabla}_L(B^L_K a + D^L_K b) + \delta^2_K a + \delta^4_K b
\end{aligned}
\tag{4.42}
$$

and the same relations with $a$ and $b$ replaced by $c$ and $d$, respectively.

A co-product on the algebra generated by $\nabla_K$ is defined by

$$\Delta(\nabla_K)(f,h)=\nabla_K(fh) \qquad (\forall f,h\in\mathscr{A}). \tag{4.43}$$

For our left-acting operators $\overleftarrow{\nabla}_K$ this takes the form

$$\begin{aligned}(f,h)\Delta(\overleftarrow{\nabla}_K)&=(fh)\overleftarrow{\nabla}_K\\ &=f\overleftarrow{\nabla}_L\Theta(h)^L_K+f(h\overleftarrow{\nabla}_K)\\ &=(f,h)(\overleftarrow{\nabla}_L\otimes\Theta^L_K+\mathbb{1}\otimes\overleftarrow{\nabla}_K)\end{aligned} \tag{4.44}$$

from which we conclude that

$$\Delta(\overleftarrow{\nabla}_K)=\overleftarrow{\nabla}_L\otimes\Theta^L_K+\mathbb{1}\otimes\overleftarrow{\nabla}_K \tag{4.45}$$

and

$$\Delta(\overleftarrow{\nabla}_K\overleftarrow{\nabla}_L)=\Delta(\overleftarrow{\nabla}_K)\Delta(\overleftarrow{\nabla}_L) \tag{4.46}$$

where

$$(f,h)\Delta(\overleftarrow{\nabla}_K\overleftarrow{\nabla}_L)=[(fh)\overleftarrow{\nabla}_K]\overleftarrow{\nabla}_L. \tag{4.47}$$

Hence, there is an interesting link between generalized derivations satisfying (4.38) and a co-product structure of the form (4.45). This can also be used in the inverse direction. Starting with a deformed Lie algebra (or some generalization), one can construct a differential calculus on the corresponding quantum group which is then defined via duality [24].

## 5. A simple differential calculus on $GL_{p,q}(2)$

In the 'classical' case we have $B=C=0$ and $A=D=\mathrm{diag}(1,1,1,1)$. A reasonable ansatz for a deformed differential calculus is therefore to keep $B=C=0$ and to require that $A$ and $D$ are diagonal. The commutation relations for $A$, $B$, $C$, $D$ are then satisfied and (4.31) restrict $A$ and $D$ to

$$A=\mathrm{diag}(\alpha,p^{-1},q^{-1},1) \qquad D=\mathrm{diag}(1,q,p,\beta) \tag{5.1}$$

with arbitrary constants $\alpha$ and $\beta$†. Evaluation of (4.36) leads to

$$\begin{aligned}\theta^2\theta^1&=-\alpha\theta^1\theta^2\\ \theta^3\theta^1&=-\alpha^{-1}\theta^1\theta^3\\ \theta^4\theta^1&=-\theta^1\theta^4+(1-r)\theta^2\theta^3\\ \theta^3\theta^2&=-r\theta^2\theta^3\\ \theta^4\theta^2&=-\beta\theta^2\theta^4\\ \theta^4\theta^3&=-\beta^{-1}\theta^3\theta^4\\ (\theta^2)^2&=(\theta^3)^2=0\\ (\theta^1)^2&=\frac{r\alpha-1}{r\alpha(\alpha+1)}\,\theta^2\theta^3\\ (\theta^4)^2&=\frac{r(r-\beta)}{\beta(\beta+1)}\,\theta^2\theta^3\end{aligned} \tag{5.2}$$

† $\alpha$ and $\beta$ have to be different from zero. Otherwise there is no 'classical limit' of the 'quantum' differential calculus.

where

$$r = pq. \tag{5.3}$$

Equation (4.16) explicitly reads

$$\begin{aligned}
\theta^1 a &= a\alpha\theta^1 & \theta^1 b &= b\theta^1 \\
\theta^2 a &= ap^{-1}\theta^2 & \theta^2 b &= bq\theta^2 \\
\theta^3 a &= aq^{-1}\theta^3 & \theta^3 b &= bp\theta^3 \\
\theta^4 &= a\theta^4 & \theta^4 b &= b\beta\theta^4
\end{aligned} \tag{5.4}$$

where the remaining equations are obtained replacing $a$ by $c$ and $b$ by $d$. Using these equations we can commute $a$ and $b$ from right to left on both sides of the third equation in (5.2). Assuming $r \neq 1$ this enforces

$$\alpha = r^{-1} \qquad \beta = r \tag{5.5}$$

so that

$$D = rA \tag{5.6}$$

and the squares of all $\theta^K$ vanish. For the left-invariant vector fields the following commutation relations result from (4.40):

$$\begin{aligned}
&\nabla_1\nabla_2 - r^{-1}\nabla_2\nabla_1 = \nabla_2 \\
&\nabla_1\nabla_3 - r\nabla_3\nabla_1 = -r\nabla_3 \\
&\nabla_1\nabla_4 - \nabla_4\nabla_1 = 0 \\
&\nabla_2\nabla_3 - r\nabla_3\nabla_2 = \nabla_1 - r\nabla_4 + (r-1)\nabla_1\nabla_4 \\
&\nabla_2\nabla_4 - r\nabla_4\nabla_2 = \nabla_2 \\
&\nabla_3\nabla_4 - r^{-1}\nabla_4\nabla_3 = -r^{-1}\nabla_3 .
\end{aligned} \tag{5.7}$$

For $r = 1$ we recover the commutation relations of the Lie algebra of $\mathrm{GL}(2, \mathbb{C})$ as expressed in the canonical basis:

$$\begin{aligned}
&\nabla_1 \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & &\nabla_2 \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&\nabla_3 \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & &\nabla_4 \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned} \tag{5.8}$$

The commutation relations between the left-invariant vector fields and elements of $\mathscr{A}$ are obtained from (4.42):

$$\begin{aligned}
\bar{\nabla}_1 a &= ar(\bar{\nabla}_1 - \mathbb{1}) & \bar{\nabla}_1 b &= b\bar{\nabla}_1 \\
\bar{\nabla}_2 a &= ap\bar{\nabla}_2 & \bar{\nabla}_2 b &= bq^{-1}\bar{\nabla}_2 - aq^{-1} \\
\bar{\nabla}_3 a &= aq\bar{\nabla}_3 - bq & \bar{\nabla}_3 b &= bp^{-1}\bar{\nabla}_3 \\
\bar{\nabla}_4 a &= a\bar{\nabla}_4 & \bar{\nabla}_4 b &= br^{-1}(\bar{\nabla}_4 - \mathbb{1})
\end{aligned} \tag{5.9}$$

(and the equations obtained by $a \to c$, $b \to d$). These are consistent with (5.7).

In order to evaluate the co-product (4.45) we need the operation of $\Theta$ on a general monomial:

$$\begin{aligned}\Theta(a^k d^l b^m c^n) &= \Theta(a)^k \Theta(d)^l \Theta(b)^m \Theta(c)^n \\ &= a^k d^l b^m c^n A^k D^l D^m A^n \\ &= a^k d^l b^m c^n r^{l+m} A^{k+l+m+n}.\end{aligned} \tag{5.10}$$

Hence

$$\Theta = r^{\mathscr{K}} A^{\mathscr{T}} \tag{5.11}$$

where the linear operators $\mathscr{K}$ and $\mathscr{T}$ count the number of $b$s and $d$s, respectively the total number of $a$, $b$, $c$, $d$s in an ordered monomial:

$$\mathscr{K} a^k d^l b^m c^n = (l+m) a^k d^l b^m c^n \tag{5.12}$$

$$\mathscr{T} a^k d^l b^m c^n = (k+l+m+n) a^k d^l b^m c^n. \tag{5.13}$$

Obviously $\mathscr{T}$ commutes with all $\bar{\nabla}_K$ and with $\mathscr{K}$. Using (5.9) we can evaluate $\bar{\nabla}_1$ and $\bar{\nabla}_4$ on a general monomial. The results are

$$\begin{aligned}\bar{\nabla}_1 &= (\mathbb{1} - r^{\mathscr{K}-\mathscr{T}})/(1-r^{-1}) \\ \bar{\nabla}_4 &= (\mathbb{1} - r^{\mathscr{K}})/(1-r).\end{aligned} \tag{5.14}$$

Inserting these expressions in (5.7), we find

$$\begin{aligned}& r^{\mathscr{K}} \bar{\nabla}_2 = \bar{\nabla}_2 r^{\mathscr{K}+1} \\ & r^{\mathscr{K}} \bar{\nabla}_3 = \bar{\nabla}_3 r^{\mathscr{K}-1} \\ & \bar{\nabla}_2 \bar{\nabla}_3 - r^{-1} \bar{\nabla}_3 \bar{\nabla}_2 = (\mathbb{1} - r^{2\mathscr{K}-\mathscr{T}})/(1-r)\end{aligned} \tag{5.15}$$

(and that $\mathscr{T}$ commutes with $\bar{\nabla}_2$, $\bar{\nabla}_3$ and $\mathscr{K}$). The first two relations lead to

$$[\mathscr{K}, \bar{\nabla}_2] = \bar{\nabla}_2 \qquad [\mathscr{K}, \bar{\nabla}_3] = -\bar{\nabla}_3. \tag{5.16}$$

Introducing

$$\mathscr{J}'_+ = \bar{\nabla}_2 r^{-\mathscr{H}/4} \qquad \mathscr{J}'_- = \bar{\nabla}_3 r^{-\mathscr{H}/4} \qquad \mathscr{H} = 2\mathscr{K} - \mathscr{T} \tag{5.17}$$

the above commutation relations are transformed to

$$[\mathscr{H}, \mathscr{J}'_\pm] = \pm 2 \mathscr{J}'_\pm \qquad [\mathscr{J}'_+, \mathscr{J}'_-] = \frac{r^{\mathscr{H}/2} - r^{-\mathscr{H}/2}}{r^{1/2} - r^{-1/2}} \qquad [\mathscr{T}, \mathscr{H}] = [\mathscr{T}, \mathscr{J}'_\pm] = 0 \tag{5.18}$$

which is a more familar deformation of the Lie algebra of the general linear group [13]. We will show that also the co-product (4.45) with (5.11) is transformed to the form given by Drinfel'd [13]. Inserting the expressions for $\bar{\nabla}_1$ and $\bar{\nabla}_4$ in (4.45), we find

$$r^{\Delta(\mathscr{K}-\mathscr{T})} = \Delta(r^{\mathscr{K}-\mathscr{T}}) = r^{\mathscr{K}-\mathscr{T}} \otimes r^{\mathscr{K}-\mathscr{T}} \qquad r^{\Delta(\mathscr{K})} = \Delta(r^{\mathscr{K}}) = r^{\mathscr{K}} \otimes r^{\mathscr{K}}. \tag{5.19}$$

Therefore

$$\Delta(\mathscr{K}) = \mathbb{1} \otimes \mathscr{K} + \mathscr{K} \otimes \mathbb{1} \qquad \Delta(\mathscr{T}) = \mathbb{1} \otimes \mathscr{T} + \mathscr{T} \otimes \mathbb{1} \tag{5.20}$$

and also

$$\Delta(\mathscr{H}) = \mathbb{1} \otimes \mathscr{H} + \mathscr{H} \otimes \mathbb{1}. \tag{5.21}$$

The commutation relations (5.18) are preserved under a rescaling of $\mathcal{J}'_+$ by some function of the operator $\mathcal{T}$ and a simultaneous rescaling of $\mathcal{J}'_-$ with the inverse function. This rescaling influences the corresponding co-product formulae. With the choice

$$\mathcal{J}_+ = \mathcal{J}'_+ p^{\mathcal{T}/4} q^{-\mathcal{T}/4} \qquad \mathcal{J}_- = \mathcal{J}'_- p^{-\mathcal{T}/4} q^{\mathcal{T}/4} \tag{5.22}$$

we obtain

$$\begin{aligned} \Delta(\mathcal{J}_+) &= \mathcal{J}_+ \otimes p^{(\mathcal{H}-\mathcal{T})/4} q^{(\mathcal{H}+\mathcal{T})/4} + p^{-(\mathcal{H}-\mathcal{T})/4} q^{-(\mathcal{H}+\mathcal{T})/4} \otimes \mathcal{J}_+ \\ \Delta(\mathcal{J}_-) &= \mathcal{J}_- \otimes p^{(\mathcal{H}+\mathcal{T})/4} q^{(\mathcal{H}-\mathcal{T})/4} + p^{-(\mathcal{H}+\mathcal{T})/4} q^{-(\mathcal{H}-\mathcal{T})/4} \otimes \mathcal{J}_- . \end{aligned} \tag{5.23}$$

Using a simpler differential calculus as in [23] we arrived at the same deformation of the Lie algebra of $GL(2, \mathbb{C})$ and the same co-product formula. With the co-unit

$$\varepsilon(\mathcal{T}) = \varepsilon(\mathcal{H}) = \varepsilon(\mathcal{J}_\pm) = 0 \qquad \varepsilon(\mathbb{1}) = 1 \tag{5.24}$$

and the antipode

$$S(\mathcal{T}) = -\mathcal{T} \qquad S(\mathcal{H}) = -\mathcal{H} \qquad S(\mathcal{J}_\pm) = -(pq)^{\mathcal{H}/4} \mathcal{J}_\pm (pq)^{-\mathcal{H}/4} \tag{5.25}$$

it becomes a Hopf algebra. There are other approaches to relate deformations of the group and deformations of the corresponding Lie algebra (see [33, 34], for example).

## 6. Bicovariant differential calculi on $GL_{p,q}(2)$

So far we only dealt with left-coaction and left-covariance. In the same way we may consider a right-coaction

$$\Delta_{\mathcal{R}}: \qquad \Lambda^1(\mathcal{A}) \to \Lambda^1(\mathcal{A}) \otimes \mathcal{A} \tag{6.1}$$

satisfying†

$$\Delta_{\mathcal{R}} \mathrm{d} = (\mathrm{d} \otimes \mathbb{1}) \Delta \tag{6.2}$$

$$\Delta_{\mathcal{R}}(f \omega h) = \Delta(f) \Delta_{\mathcal{R}}(\omega) \Delta(h) \qquad \forall f, h \in \mathcal{A}, \omega \in \Lambda^1(\mathcal{A}). \tag{6.3}$$

A 1-form $\omega$ is right-coinvariant if

$$\Delta_{\mathcal{R}} \omega = \omega \otimes \mathbb{1}. \tag{6.4}$$

Right-coinvariant Maurer–Cartan 1-forms on $GL_{p,q}(2)$ are given by

$$\begin{pmatrix} \omega^1 & \omega^2 \\ \omega^3 & \omega^4 \end{pmatrix} = \mathrm{d} \begin{pmatrix} a & b \\ c & d \end{pmatrix} S \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{6.5}$$

The left-coinvariant 1-forms $\theta^K$ and the right-coinvariant $\omega^K$ are related by

$$\theta = S \begin{pmatrix} a & b \\ c & d \end{pmatrix} \omega \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6.6}$$

$$\omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \theta S \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{6.7}$$

† (6.2) is only needed to demonstrate the right-coinvariance of (6.5).

Now we calculate the right coaction on $\theta^K$ by expressing them in terms of the $\omega^K$, using the homomorphism property of the right-coaction, the right-coinvariance of the $\omega^K$, and translating the latter back into the $\theta^K$. The result is

$$\Delta_{\mathscr{R}}(\theta^I) = \theta^J \otimes R_J{}^I \tag{6.8}$$

with

$$\begin{aligned}(R_J{}^I) &= \begin{pmatrix} S(a) & S(c) \\ S(b) & S(d) \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} S(a)a & S(a)b & S(c)a & S(c)b \\ S(a)c & S(a)d & S(c)c & S(c)d \\ S(b)a & S(b)b & S(d)a & S(d)b \\ S(b)c & S(b)d & S(d)c & S(d)d \end{pmatrix}. \end{aligned} \tag{6.9}$$

The matrix $R$ (with entries in $\mathscr{A}$) has the properties (see also theorem 2.4 in [20])

$$\Delta(R_J{}^I) = R_J{}^K \otimes R_K{}^I \tag{6.10}$$

$$\varepsilon(R_J{}^I) = \delta_J^I \tag{6.11}$$

$$S(R_J{}^K)R_K{}^I = \delta_J^I = R_J{}^K S(R_K{}^I) \tag{6.12}$$

where we have used (2.16) to derive the last equation.

The existence of a 'bi-coaction' [20], i.e. a left- *and* a right-coaction, imposes restrictions on the differential calculus, more precisely: on the commutation relations between left-coinvariant 1-forms and elements of $\mathscr{A}$. This is seen as follows. We have

$$\Delta_{\mathscr{R}}(\theta^I f) = \Delta_{\mathscr{R}}(\theta^I)\Delta(f) = (\theta^K \otimes R_K{}^I)\Delta(f)$$

which has to be the same as

$$\Delta_{\mathscr{R}}(\Theta(f)^I_J\theta^J) = \Delta(\Theta(f)^I_J)\Delta_{\mathscr{R}}(\theta^J) = \Delta(\Theta(f)^I_J)(\theta^K \otimes R_K{}^J).$$

Hence

$$(\theta^K \otimes R_K{}^I)\Delta(f) = \Delta(\Theta(f)^I_J)(\theta^K \otimes R_K{}^J) \qquad \forall f \in \mathscr{A}. \tag{6.13}$$

Applied to $a$, $b$, $c$, $d$ this leads to the following equations† constraining the matrices $A$, $B$, $C$ and $D$:

$$R_K{}^I(A_J^K a + B_J^K c) = (aA_K^I + bC_K^I)R_J{}^K \tag{6.14}$$

$$R_K{}^I(C_J^K a + D_J^K c) = (cA_K^I + dC_K^I)R_J{}^K \tag{6.15}$$

$$R_K{}^I(A_J^K b + B_J^K d) = (aB_K^I + bD_K^I)R_J{}^K \tag{6.16}$$

$$R_K{}^I(C_J^K b + D_J^K d) = (cB_K^I + dD_K^I)R_J{}^K. \tag{6.17}$$

*Remark.* Using $\mathscr{F}$ to denote the representation of the algebra given by the matrices $A$, $B$, $C$, $D$ (as in a previous remark), these equations can be written as

$$R_K{}^I(\mathscr{F}^K{}_J \otimes \mathbb{1})\Delta(f) = (\mathbb{1} \otimes \mathscr{F}^I{}_K)\Delta(f)R_J{}^K \tag{6.18}$$

with $f = a, b, c, d$. In terms of convolution products this reads

$$R_K{}^I(f \star \mathscr{F}^K{}_J) = (\mathscr{F}^I{}_K \star f)R_J{}^K \tag{6.19}$$

(cf equation (2.39) in [20]). □

† It is actually sufficient to evaluate this equation on $a$, $b$, $c$, $d$ since the homomorphism properties of $\Delta$ and $\Theta$ then ensure that it also holds for an arbitrary $f \in \mathscr{A}$.

Inserting the result (6.9) for the matrix $R$, using the commutation relations for $a$, $b$, $c$, $d$, $\mathscr{D}^{-1}$ and taking (4.31) into account, the most general solution of these equations is given by

$$A=\begin{pmatrix} A_1^1 & 0 & 0 & A_4^1 \\ 0 & q\beta & 0 & 0 \\ 0 & 0 & p\beta & 0 \\ A_1^4 & 0 & 0 & A_4^4 \end{pmatrix} \qquad B=\begin{pmatrix} 0 & B_2^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ B_1^3 & 0 & 0 & B_4^3 \\ 0 & B_2^4 & 0 & 0 \end{pmatrix}$$
$$C=\begin{pmatrix} 0 & 0 & B_2^1 & 0 \\ (q/p)B_1^3 & 0 & 0 & (q/p)B_4^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & B_2^4 & 0 \end{pmatrix} \qquad D=\begin{pmatrix} D_1^1 & 0 & 0 & D_4^1 \\ 0 & q\beta & 0 & 0 \\ 0 & 0 & p\beta & 0 \\ rA_4^1 & 0 & 0 & D_4^4 \end{pmatrix} \tag{6.20}$$

where

$$\begin{aligned}
&\beta=(1+A_1^1-rA_4^1)/(1+r)\\
&A_1^4=rD_1^1-[2r+(r-1)A_1^1+r^2(r-1)A_4^1]/(1+r)\\
&A_4^4=D_1^1+(1-r)A_4^1\\
&B_2^1=-(1-rA_1^1+r^2A_4^1)/(1+r)\\
&B_1^3=p[(1+A_1^1+r^2A_4^1)/(1+r)-D_1^1]\\
&B_4^3=\frac{1}{q}[r(1+A_1^1+A_4^1)/(1+r)-D_1^1]\\
&B_2^4=-(r-A_1^1+rA_4^1)/(1+r)\\
&D_4^1=\frac{1}{r}D_1^1-[2+(1-r)A_1^1+r(r-1)A_4^1]/(1+r)\\
&D_4^4=A_1^1+(1-r)A_4^1
\end{aligned} \tag{6.21}$$

and $r=pq$. In addition,the matrices $A$, $B$, $C$, $D$ have to satisfy the same commutation relations as $a$, $b$, $c$, $d$. This leads to the following equations:

$$\begin{aligned}
&B_2^1B_1^3+B_4^3B_2^4=0\\
&(A_1^1-r\beta)B_2^1+A_4^1B_2^4=0\\
&(A_1^1-\beta)B_1^3+A_1^4B_4^3=0\\
&(A_4^4-\beta)B_4^3+A_4^1B_1^3=0\\
&(A_4^4-r\beta)B_2^4+A_1^4B_2^1=0\\
&(D_1^1-\beta)B_2^1+D_4^1B_2^4=0\\
&(D_1^1-r\beta)B_1^3+rA_4^1B_4^3=0\\
&(D_4^4-r\beta)B_4^3+D_4^1B_1^3=0\\
&(D_4^4-\beta)B_2^4+rA_4^1B_2^1=0
\end{aligned} \tag{6.22}$$

and

$$r(A^1_4)^2 - A^4_1 D^1_4 + \frac{1-r}{p} B^1_2 B^3_1 = 0$$

$$(A^1_1 - A^4_4) D^1_4 - (D^1_1 - D^4_4) A^1_4 + \frac{1-r}{p} B^1_2 B^3_4 = 0 \tag{6.23}$$

$$(A^1_1 - A^4_4) r A^1_4 - (D^1_1 - D^4_4) A^4_1 - \frac{1-r}{p} B^3_1 B^4_2 = 0.$$

As a consequence of (6.22),

$$B^1_2 B^3_4 [A^1_1 + A^4_4 - (1+r)\beta] = 0$$

$$B^3_1 B^4_2 [A^1_1 + A^4_4 - (1+r)\beta] = 0.$$

If $A^1_1 + A^4_4 - (1+r)\beta \neq 0$ (and if we require a classical limit) we are forced to set $r = 1$. This case still deserves further discussion and will be treated at the end of section 9. But in the following we concentrate on the case where the deformation parameters $p$ and $q$ are not constrained. Let us therefore turn to the complementary case $A^1_1 + A^4_4 - (1+r)\beta = 0$. This means

$$D^1_1 = 1 - A^1_4. \tag{6.24}$$

Surprisingly, all the remaining equations (6.22) and (6.23) now reduce to the single equation

$$(A^1_1)^2 - \frac{1}{r}[1 + r^2 - (1 + r - r^2 + r^3) A^1_4] A^1_1$$
$$+ 1 - (1 + 2r - r^2) A^1_4 - (1 + r + r^3)(A^1_4)^2 = 0. \tag{6.25}$$

We have arrived at our main result:

The most general bicovariant first-order differential calculus on $GL_{p,q}(2)$ (possessing a classical limit) has two branches depending on one additional parameter $s := A^1_4$ (if we understand the last equation to be solved for $A^1_1$)†.

*Remark.* From theorem 2.5 in [20] we conclude that the differential calculus with (6.20), (6.21), (6.24) and (6.25) defines a 'bicovariant bimodule' [20]. A first order bicovariant differential calculus always admits a consistent extension to higher order (2-forms etc), an important result due to Woronowicz [20] (see section 7 for some details). □

For the special parameter value $s = 0$ the two solutions are listed in the following.

† There is one further solution depending on $s$ if we restrict $p$ and $q$ by $p = 1/q$, see section 9.

*Example 1.*

$$A=\begin{pmatrix} r & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad B=\begin{pmatrix} 0 & r-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (r-1)/q \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$C=\begin{pmatrix} 0 & 0 & r-1 & 0 \\ 0 & 0 & 0 & (r-1)/p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad D=\begin{pmatrix} 1 & 0 & 0 & (r-1)^2/r \\ 0 & q & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & r \end{pmatrix}. \tag{6.26}$$

From (4.36) we obtain the commutation relations for the left-coinvariant Maurer-Cartan forms:

$$\begin{aligned}
&(\theta^1)^2=\frac{1-r}{r}\,\theta^2\theta^3 \\
&\theta^2\theta^1=-r\theta^1\theta^2+(1-r)\theta^2\theta^4 \\
&\theta^3\theta^1=-\frac{1}{r}\,\theta^1\theta^3+\frac{r-1}{r}\,\theta^3\theta^4 \\
&\theta^4\theta^1=-\theta^1\theta^4+(r-1)\theta^2\theta^3 \\
&\theta^3\theta^2=-\theta^2\theta^3 \\
&\theta^4\theta^2=-\theta^2\theta^4 \\
&\theta^4\theta^3=-\theta^3\theta^4 \\
&(\theta^2)^2=(\theta^3)^2=(\theta^4)^2=0.
\end{aligned} \tag{6.27}$$

Commuting $a$, $b$, $c$, $d$ through these relations does not lead to additional restrictions. The algebra of left-coinvariant vector fields is

$$\begin{aligned}
&\bar\nabla_2\bar\nabla_1-r\bar\nabla_1\bar\nabla_2=\bar\nabla_2 \\
&\bar\nabla_3\bar\nabla_1-\frac{1}{r}\bar\nabla_1\bar\nabla_3=-\frac{1}{r}\bar\nabla_3 \\
&\bar\nabla_4\bar\nabla_1-\bar\nabla_1\bar\nabla_4=0 \\
&\bar\nabla_3\bar\nabla_2-\bar\nabla_2\bar\nabla_3=-\bar\nabla_4+\frac{1}{r}\bar\nabla_1+(1-r)\bar\nabla_1\bar\nabla_4+\frac{r-1}{r}\bar\nabla_1^2 \\
&\bar\nabla_4\bar\nabla_2-\bar\nabla_2\bar\nabla_4=\bar\nabla_2+(r-1)\bar\nabla_1\bar\nabla_2 \\
&\bar\nabla_4\bar\nabla_3-\bar\nabla_3\bar\nabla_4=-\frac{1}{r}\bar\nabla_3+\frac{1-r}{r}\bar\nabla_1\bar\nabla_3.
\end{aligned} \tag{6.28}$$

This is consistent with the commutation relations (4.42) between $\bar\nabla_K$ and elements of $\mathscr{A}$.

*Example 2.*

$$A=\begin{pmatrix} 1/r & 0 & 0 & 0 \\ 0 & 1/p & 0 & 0 \\ 0 & 0 & 1/q & 0 \\ (1-r)^2/r & 0 & 0 & 1 \end{pmatrix} \qquad B=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (1-r)/q & 0 & 0 & 0 \\ 0 & (1-r)/r & 0 & 0 \end{pmatrix}$$
$$C=\begin{pmatrix} 0 & 0 & 0 & 0 \\ (1-r)/p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (1-r)/r & 0 \end{pmatrix} \qquad D=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/p & 0 & 0 \\ 0 & 0 & 1/q & 0 \\ 0 & 0 & 0 & 1/r \end{pmatrix}. \tag{6.29}$$

From (4.36) we obtain

$$\begin{aligned}
&(\theta^1)^2=(\theta^2)^2=(\theta^3)^2=0 \\
&\theta^2\theta^1=-\theta^1\theta^2 \\
&\theta^3\theta^1=-\theta^1\theta^3 \\
&\theta^4\theta^1=-\theta^1\theta^4+\frac{r-1}{r}\,\theta^2\theta^3 \\
&\theta^3\theta^2=-\theta^2\theta^3 \\
&\theta^4\theta^2=-\frac{1}{r}\,\theta^2\theta^4+\frac{r-1}{r}\,\theta^1\theta^2 \\
&\theta^4\theta^3=-r\theta^3\theta^4+(1-r)\theta^1\theta^3 \\
&(\theta^4)^2=(1-r)\theta^2\theta^3.
\end{aligned} \tag{6.30}$$

Again, these equations are consistent with the commutation relations between $\theta^K$ and elements of $\mathscr{A}$. The algebra of left-coinvariant vector fields is

$$\begin{aligned}
&\bar{\nabla}_2\bar{\nabla}_1-\bar{\nabla}_1\bar{\nabla}_2=\bar{\nabla}_2+\frac{1-r}{r}\bar{\nabla}_2\bar{\nabla}_4 \\
&\bar{\nabla}_3\bar{\nabla}_1-\bar{\nabla}_1\bar{\nabla}_3=-r\bar{\nabla}_3+(r-1)\bar{\nabla}_3\bar{\nabla}_4 \\
&\bar{\nabla}_4\bar{\nabla}_1-\bar{\nabla}_1\bar{\nabla}_4=0 \\
&\bar{\nabla}_3\bar{\nabla}_2-\bar{\nabla}_2\bar{\nabla}_3=\bar{\nabla}_1-r\bar{\nabla}_4+\frac{1-r}{r}\bar{\nabla}_1\bar{\nabla}_4+(r-1)\bar{\nabla}_4^2 \\
&\bar{\nabla}_4\bar{\nabla}_2-\frac{1}{r}\bar{\nabla}_2\bar{\nabla}_4=\bar{\nabla}_2 \\
&\bar{\nabla}_4\bar{\nabla}_3-r\bar{\nabla}_3\bar{\nabla}_4=-r\bar{\nabla}_3.
\end{aligned} \tag{6.31}$$

Again, this is consistent with the commutation relations (4.42) between $\bar{\nabla}_K$ and elements of $\mathscr{A}$.

Relations (6.29) and (6.30) are equivalent to the commutation relations of example 7 in [29] (in the case where the 'monoïde quantique' corresponds to a 'déformation de l'objet commutatif'). There one can find the commutation relations in terms of the differentials of $a$, $b$, $c$, $d$ (instead of the Maurer–Cartan 1-forms $\theta^K$)†. There is a generalization of this differential calculus to quantum deformations of GL($n$) [30].

† In order to compare the formulae in [29] with ours one has to replace $q$ by $1/q$ and $p$ by $1/p$.

Our general result also shows that the simple left-covariant differential calculus considered in section 5 is not bicovariant.

In order to calculate the $\theta^K$ commutation relations (i.e. the deformed wedge product) for the general bicovariant calculus, (4.36) appears to be rather intractable. Fortunately, there is a trickier way to do it. After some preparations in the following section we will return to this problem in section 8.

## 7. Higher-order forms in bicovariant differential calculus

In this section we recall the essential steps of Woronowicz' important result that a bicovariant first-order differential calculus always admits a unique extension to higher-order forms [20]. All the results of this section (and many more) are contained in his work. We have merely arranged them in a different way, concentrating on what we need in the following and illustrating some points with examples.

There is a kind of inverse formula of (4.16):

$$f\theta^I = \theta^J \hat{\Theta}(f)^I_J. \tag{7.1}$$

Applying (4.16), we find

$$\hat{\Theta}^K_J(\Theta(f)^I_K) = f\delta^I_J \qquad \forall f \in \mathcal{A} \tag{7.2}$$

and therefore

$$\begin{aligned}\hat{\Theta}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}(AD - pBC)^{-1}\begin{pmatrix} D & -pB \\ -C/p & A \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathcal{F}\circ S^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\end{aligned} \tag{7.3}$$

with the notation of section 4. Then

$$f\theta^I = \theta^J(\mathbb{1}\otimes \mathcal{F}^I{}_J \circ S^{-1})\Delta(f) = \theta^J(\mathcal{F}^I{}_J \circ S^{-1} \star f) \tag{7.4}$$

(cf (2.14) in [20]).

Let us introduce [20]

$$\tilde{\omega}^K = \theta^L S(R_L{}^K). \tag{7.5}$$

These 1-forms are right-coinvariant. To verify this, one needs the identity

$$\Delta(S(R_I{}^J)) = S(R_K{}^J) \otimes S(R_I{}^K) \tag{7.6}$$

which follows from (6.12) by application of $\Delta$. The 1-forms $\tilde{\omega}^K$ are *not identical with* the $\omega^K$ defined in (6.5) (or (6.7)). But since they are also right-coinvariant, they must be a linear combination of the $\omega^K$ with coefficients in $\mathbb{C}$. For the bicovariant calculus in example 1 of section 6 one finds

$$\begin{aligned}\tilde{\omega}^1 &= r\omega^1 + r(r-1)\omega^4 \\ \tilde{\omega}^2 &= q\omega^2 \\ \tilde{\omega}^3 &= (p^2/q)\omega^3 \\ \tilde{\omega}^4 &= r^2\omega^4.\end{aligned} \tag{7.7}$$

Furthermore,

$$\omega^1-\tilde{\omega}^1=\frac{1-r}{r}\left(\theta^1+\frac{1}{r}\theta^4\right). \tag{7.8}$$

The LHS is right-coinvariant and the RHS is manifestly left-coinvariant. Therefore

$$\rho=\theta^1+\theta^4/r \tag{7.9}$$

is a 'bi-coinvariant' 1-form:

$$\Delta_{\mathscr{L}}(\rho)=\mathbb{1}\otimes\rho \qquad \Delta_{\mathscr{R}}(\rho)=\rho\otimes\mathbb{1}. \tag{7.10}$$

Using (7.5) and (6.10) the condition of bi-coinvariance for a 1-form $\rho=\rho_K\theta^K$ translates to

$$R_K{}^L\rho_L=\rho_K \qquad \rho_K\in\mathbb{C}. \tag{7.11}$$

It is easy to show that $\theta^1+\theta^4/r$ is (up to multiplication by a complex number) the only bi-coinvariant 1-form on $\mathrm{GL}_{p,q}(2)$. This does not depend on the choice of bicovariant differential calculus.

*The next result is the analogue of (7.4) for the right-coinvariant 1-forms $\tilde{\omega}^K$.*

*Lemma.* For a bicovariant differential calculus,

$$f\tilde{\omega}^K=\tilde{\omega}^L(f\star\mathscr{F}^K{}_L\circ S^{-1}) \qquad (\forall f\in\mathscr{A}). \tag{7.12}$$

*Proof.* First we note that

$$\Delta\circ S^{-1}=\sigma(S^{-1}\otimes S^{-1})\Delta$$

(see also proposition 1.9 in [35]) where $\sigma$ is the flip automorphism

$$\sigma(f\otimes h)=h\otimes f \qquad \forall f,h\in\mathscr{A}.$$

Then

$$\begin{aligned}(S^{-1}f)\star\mathscr{F}^I{}_J&=(\mathscr{F}^I{}_J\otimes\mathbb{1})\Delta(S^{-1}f)\\&=(\mathscr{F}^I{}_J\otimes\mathbb{1})\sigma(S^{-1}\otimes S^{-1})\Delta(f)\\&=\sigma\underbrace{((S^{-1}\otimes\mathscr{F}^I{}_J\circ S^{-1})\Delta(f))}_{\in\mathscr{A}\otimes\mathbb{C}=\mathscr{A}}\\&=S^{-1}((\mathbb{1}\otimes\mathscr{F}^I{}_J\circ S^{-1})\Delta(f)\\&=S^{-1}(\mathscr{F}^I{}_J\circ S^{-1}\star f)\end{aligned}$$

and similarly

$$\mathscr{F}^I{}_J\star(S^{-1}f)=S^{-1}(f\star\mathscr{F}^I{}_J\circ S^{-1}).$$

Using these identities in the bicovariance condition (6.19) and acting with $S$ on the resulting equation yields

$$(\mathscr{F}^K{}_J\circ S^{-1}\star f)S(R_K{}^I)=S(R_J{}^K)(f\star\mathscr{F}^I{}_K\circ S^{-1}).$$

As a consequence,

$$\begin{aligned}f\tilde{\omega}^K&=f\theta^LS(R_L{}^K)\\&=\theta^M(\mathscr{F}^L{}_M\circ S^{-1}\star f)S(R_L{}^K)\\&=\theta^MS(R_M{}^L)(f\star\mathscr{F}^K{}_L\circ S^{-1})\\&=\tilde{\omega}^L(f\star\mathscr{F}^K{}_L\circ S^{-1})\end{aligned}$$

which is the asserted formula. □

The corresponding inverse formula is [20]

$$\tilde{\omega}^K f = (f \star \mathcal{F}^K{}_L)\tilde{\omega}^L := \tilde{\Theta}(f)^K_L \tilde{\omega}^L \tag{7.13}$$

where

$$\tilde{\Theta}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{7.14}$$

Classically 2-forms are introduced by antisymmetrization of tensor products of 1-forms. To do something similar in the non-commutative case, one needs an analogue of the (anti-)symmetrization operation. Any element $\tau \in \Lambda^1(\mathcal{A}) \otimes_{\mathcal{A}} \Lambda^1(\mathcal{A})$ can be written as†

$$\tau = \tau_{IJ}\theta^I \otimes_{\mathcal{A}} \tilde{\omega}^J \tag{7.15}$$

with $\tau_{IJ} \in \mathcal{A}$. Define [20]

$$\sigma: \qquad \Lambda^1(\mathcal{A}) \otimes_{\mathcal{A}} \Lambda^1(\mathcal{A}) \to \Lambda^1(\mathcal{A}) \otimes_{\mathcal{A}} \Lambda^1(\mathcal{A})$$

by

$$\sigma(\tau) = \tau_{IJ}\sigma(\theta^I \otimes_{\mathcal{A}} \tilde{\omega}^J) = \tau_{IJ}\tilde{\omega}^J \otimes_{\mathcal{A}} \theta^I. \tag{7.16}$$

Then

$$\sigma(f\tau) = f\sigma(\tau) \qquad \forall f \in \mathcal{A}. \tag{7.17}$$

But we also have

$$\sigma(\tau f) = \sigma(\tau) f \tag{7.18}$$

since

$$\begin{aligned} \sigma(\theta^K \otimes_{\mathcal{A}} \tilde{\omega}^L f) &= \sigma(\theta^K \otimes_{\mathcal{A}} (f \star \mathcal{F}^L{}_M)\tilde{\omega}^M) \\ &= \sigma(\theta^K (f \star \mathcal{F}^L{}_M \otimes_{\mathcal{A}} \tilde{\omega}^M) \\ &= \sigma(\mathcal{F}^K{}_N \star (f \star \mathcal{F}^L{}_M)\theta^N \otimes_{\mathcal{A}} \tilde{\omega}^M) \\ &= (\mathcal{F}^K{}_N \bullet f) \bullet \mathcal{F}^L{}_M \tilde{\omega}^M \otimes_{\mathcal{A}} \theta^N \\ &= \tilde{\omega}^L (\mathcal{F}^K{}_N \star f) \otimes_{\mathcal{A}} \theta^N \\ &= \tilde{\omega}^L \otimes_{\mathcal{A}} (\mathcal{F}^K{}_N \star f)\theta^N \\ &= \tilde{\omega}^L \otimes_{\mathcal{A}} \theta^K f \\ &= \sigma(\theta^K \otimes_{\mathcal{A}} \tilde{\omega}^L) f \end{aligned} \tag{7.19}$$

where we used the associativity of the convolution product. $\sigma$ is therefore a bimodule *homomorphism* (and moreover an isomorphism). It replaces and reduces to the permutation operation in the classical case. The exterior product of, for example, left-coinvariant Maurer–Cartan 1-forms is now defined by

$$\theta^I \theta^J \equiv \theta^I \wedge_{\mathcal{A}} \theta^J = (\mathbb{1} - \sigma)\theta^I \otimes_{\mathcal{A}} \theta^J \tag{7.20}$$

and this extends in an obvious way to higher-order forms. $\sigma$ has the property

$$\sigma(\theta \otimes_{\mathcal{A}} \omega) = \omega \otimes_{\mathcal{A}} \theta \tag{7.21}$$

† The tensor product over $\mathcal{A}$ means that we identify $\theta \otimes_{\mathcal{A}} f\omega$ with $\theta f \otimes_{\mathcal{A}} \omega$ for all 1-forms $\theta$, $\omega$ and $f \in \mathcal{A}$. For 2-forms this property was already presumed in writing them as, for example, $\theta^K \theta^L$.

for any left-coinvariant 1-form $\theta$ and any right-coinvariant 1-form $\omega$. Applied to the bi-coinvariant 1-form $\rho$ introduced in (7.9) this leads to

$$0=\rho\wedge_{\mathscr{A}}\rho=(\theta^1)^2+\frac{1}{r^2}(\theta^4)^2+\frac{1}{r}(\theta^1\theta^4+\theta^4\theta^1) \tag{7.22}$$

and this holds for *all* bicovariant calculi on $GL_{p,q}(2)$.

## 8. More on bicovariant differential calculus

Another interesting result obtained by Woronowicz (cf the proof of theorem 4.1 in [20]) is that the exterior derivative of a bicovariant differential calculus can be expressed as

$$\mathrm{d}f=\frac{1}{N}[\rho,f]_- \qquad (\forall f\in\mathscr{A}) \tag{8.1}$$

$$\mathrm{d}\theta=\frac{1}{N}[\rho,\theta]_+ \qquad (\forall\theta\in\Lambda^1(\mathscr{A})) \tag{8.2}$$

(and correspondingly for higher forms) where $\rho$ is a bi-coinvariant element of the (possibly to be extended) space of 1-forms. The 1-form $\rho$ given in (7.9) is therefore a candidate. The Leibniz rule is then automatically satisfied and we have $\mathrm{d}^2=0$ as a consequence of (7.22). Using (4.12) we find that (8.1) holds with

$$N=\frac{1}{r(r+1)}[(r^2+1)(A^1_1-rs)-2r] \tag{8.3}$$

for the general bicovariant differential calculus in section 6 ($A^1_1$ and $s=A^1_4$ are subject to equation (6.25)). In particular, we have $N=r-1$ and $N=(1-r)/r^2$, respectively, for the two bicovariant calculi given explicitly in section 6. The 'normalization constant' $N$ thus depends on the respective bicovariant calculus.

Equation (8.2) and the Maurer–Cartan equation (4.32) yield

$$\left[\theta^1+\frac{1}{r}\theta^4,\theta^K\right]_+=-NC^K{}_{MN}\theta^M\theta^N \tag{8.4}$$

and this equation determines some commutation relations between the Maurer–Cartan 1-forms $\theta^K$. Together with (7.22) and (4.36) a lengthy calculation leads to†

$$(\theta^1)^2=\frac{1-r(1+rN)}{r+r^2(1+N)}\theta^2\theta^3$$

$$(\theta^2)^2=(\theta^3)^2=0$$

$$(\theta^4)^2=\frac{r(1-r+N)}{1+r(1+N)}\theta^2\theta^3$$

† The $\theta^3\theta^1$ term (with the 'wrong' ordering) in the last equation is not a misprint. We just wanted to avoid a lengthier expression which arises by reordering it.

$$\theta^2\theta^1=\frac{1}{1+r(1+N)}(-r[2+N(1+r-s(1+r^2))]\theta^1\theta^2$$
$$+[1-r-rN(r-s(1+r^2))]\theta^2\theta^4)$$
$$\theta^3\theta^1=\frac{1}{M}(-[2+N(1+r-s(1+r^2))]\theta^1\theta^3$$
$$+[r-1+rN(r-s(1+r^2))]\theta^3\theta^4) \tag{8.5}$$
$$\theta^4\theta^1=-\theta^1\theta^4+(r-1)\frac{2+(r+1)N}{1+r(1+N)}\theta^2\theta^3$$
$$\theta^3\theta^2=-\theta^2\theta^3$$
$$\theta^4\theta^2=\frac{1}{1+r(1+N)}(-[2+N(1+r-s(1+r^2))]\theta^2\theta^4$$
$$+[r-1-N(1-s(1+r^2))]\theta^1\theta^2)$$
$$\theta^4\theta^3=-\frac{1}{1+rN}(\theta^3\theta^4+r\theta^1\theta^3+r(1+N)\theta^3\theta^1)$$

where

$$M=1+r+N[r^2+r+1-s(r^3+r^2+r+1)]. \tag{8.6}$$

These relations are consistent with the commutation relations between $\theta^K$ and elements of $\mathscr{A}$. The corresponding left-coinvariant vector fields satisfy the following algebra:

$$\bar{\nabla}_2\bar{\nabla}_1=\frac{r}{1+r(1+N)}[2+N(1+r-s(1+r^2))]\bar{\nabla}_1\bar{\nabla}_2+\bar{\nabla}_2$$
$$+\frac{1-r+N(1-s(1+r^2))}{1+r(1+N)}\bar{\nabla}_2\bar{\nabla}_4$$
$$\bar{\nabla}_4\bar{\nabla}_1=\bar{\nabla}_1\bar{\nabla}_4$$
$$\bar{\nabla}_3\bar{\nabla}_2=\bar{\nabla}_2\bar{\nabla}_3+(1-r)\frac{2+N(1+r)}{1+r(1+N)}\bar{\nabla}_1\bar{\nabla}_4+\frac{1+r^2}{1+r(1+N)}\left(\frac{1}{r}\bar{\nabla}_1-\bar{\nabla}_4\right)$$
$$-\frac{1-r(1+rN)}{r+r^2(1+N)}\bar{\nabla}_1^2-\frac{r(1-r+N)}{1+r(1+N)}\bar{\nabla}_4^2$$
$$\bar{\nabla}_4\bar{\nabla}_2=\frac{2+N(1+r-s(1+r^2))}{1+r(1+N)}\bar{\nabla}_2\bar{\nabla}_4+\bar{\nabla}_2-\frac{1-r+rN(s(1+r^2)-r)}{1+r(1+N)}\bar{\nabla}_1\bar{\nabla}_2$$
$$\bar{\nabla}_3\bar{\nabla}_1=\frac{1}{M}[2+N(1+r-s(1+r^2))]\bar{\nabla}_1\bar{\nabla}_3 \tag{8.7}$$
$$-\frac{1}{M(1+rN)}\{2+r(r-1)+N[1+r+r^3-s(1+r^2)^2]\}\bar{\nabla}_3$$
$$-\frac{r}{M(1+rN)}[1-r+N(2-r^2+rs(1+r^2))$$
$$+N^2(1+r-s(1+r^2))]\bar{\nabla}_3\bar{\nabla}_4$$

$$\bar{\nabla}_4\bar{\nabla}_3 = \frac{1}{M(1+rN)}\{1+r^2+N[1+2r^2+r^3-s(1+r+2r^2+r^3+r^4)]$$

$$+r^2N^2[r-s(1+r^2)]\}\bar{\nabla}_3\bar{\nabla}_4 + \frac{1}{M}[1-r+rN(s(1+r^2)-r)]\bar{\nabla}_1\bar{\nabla}_3$$

$$-\frac{1}{M(1+rN)}\{2-r+r^2+N[1+r+r^3-s(1+r^2)^2]\}\bar{\nabla}_3.$$

Besides $r=pq$ this algebra depends on the new parameter $s$. Note that $N$ is a function of $s$ via (8.3) and (6.25). The commutation relations for the vector fields are consistent with (4.42).

It should still be investigated whether the above algebra is a Hopf algebra. However, the corresponding calculation is not simple. In particular, the operator $\Theta$ appearing in the co-product formula (4.45) has to be expressed in terms of the left-coinvariant vector fields (see the corresponding calculation in section 5).

## 9. Non-standard bicovariant calculi on GL(2)

In this section we extract some simple, but interesting examples from the general results of the sections 6 and 8. This concerns deformed differential calculi on the classical general linear group GL(2). Actually we will only set

$$r=1 \tag{9.1}$$

i.e. $p=1/q$, and therefore still allow a deformation of GL(2). As most interesting, however, we regard the deformations which are even present in the commutative case where also $q=1$. With the above restriction, there are two solutions of (6.25). The first is

$$A_1^1 = 1-3s \tag{9.2}$$

with $N=-4s$ and the wedge product is given by

$$\begin{aligned}
&(\theta^1)^2 = \frac{\tilde{s}}{1-\tilde{s}}\,\theta^2\theta^3 \qquad (\theta^2)^2=(\theta^3)^2=0 \qquad (\theta^4)^2 = -\frac{\tilde{s}}{1-\tilde{s}}\,\theta^2\theta^3\\
&\theta^2\theta^1 = (\tilde{s}-1)\,\theta^1\theta^2+\tilde{s}\theta^2\theta^4\\
&\theta^3\theta^1 = \frac{1}{2\tilde{s}-1}[(1-\tilde{s})\theta^1\theta^3+\tilde{s}\theta^3\theta^4]\\
&\theta^4\theta^1 = -\theta^1\theta^4 \qquad \theta^3\theta^2 = -\theta^2\theta^3\\
&\theta^4\theta^2 = (\tilde{s}-1)\,\theta^2\theta^4+\tilde{s}\theta^1\theta^2\\
&\theta^4\theta^3 = \frac{1}{2\tilde{s}-1}[(1-\tilde{s})\theta^3\theta^4+\tilde{s}\theta^1\theta^3]
\end{aligned} \tag{9.3}$$

with $\tilde{s} := 2s$. The algebra of vector fields is

$$
\begin{aligned}
&\bar{\nabla}_2\bar{\nabla}_1 = (1-\tilde{s})\bar{\nabla}_1\bar{\nabla}_2 + \bar{\nabla}_2 - \tilde{s}\bar{\nabla}_2\bar{\nabla}_4 \\
&\bar{\nabla}_3\bar{\nabla}_1 = \frac{1}{1-2\tilde{s}}((1-\tilde{s})\bar{\nabla}_1\bar{\nabla}_3 + \tilde{s}\bar{\nabla}_3\bar{\nabla}_4) \\
&\bar{\nabla}_4\bar{\nabla}_1 = \bar{\nabla}_1\bar{\nabla}_4 \\
&\bar{\nabla}_3\bar{\nabla}_2 = \bar{\nabla}_2\bar{\nabla}_3 + \frac{1}{1-\tilde{s}}(\bar{\nabla}_1 - \bar{\nabla}_4 - \tilde{s}(\bar{\nabla}_1^2 - \bar{\nabla}_4^2)) \\
&\bar{\nabla}_4\bar{\nabla}_2 = (1-\tilde{s})\bar{\nabla}_2\bar{\nabla}_4 - \tilde{s}\bar{\nabla}_1\bar{\nabla}_2 + \bar{\nabla}_2 \\
&\bar{\nabla}_4\bar{\nabla}_3 = \frac{1}{1-2\tilde{s}}((1-\tilde{s})\bar{\nabla}_3\bar{\nabla}_4 + \tilde{s}\bar{\nabla}_1\bar{\nabla}_3 - \bar{\nabla}_3).
\end{aligned}
\tag{9.4}
$$

The matrix representation of the algebra $\mathscr{A}$ is in this case

$$
\begin{aligned}
&A = \begin{pmatrix} 1-3s & 0 & 0 & s \\ 0 & q(1-2s) & 0 & 0 \\ 0 & 0 & (1-2s)/q & 0 \\ -s & 0 & 0 & 1-s \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -2s & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2s & 0 & 0 \end{pmatrix} \\
&C = \begin{pmatrix} 0 & 0 & -2s & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2s & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1-s & 0 & 0 & -s \\ 0 & q(1-2s) & 0 & 0 \\ 0 & 0 & (1-2s)/q & 0 \\ s & 0 & 0 & 1-3s \end{pmatrix}.
\end{aligned}
\tag{9.5}
$$

The second solution is

$$
A^1_1 = 1 + s \tag{9.6}
$$

which leads to† $N = 0$ and the wedge product turns out to be the classical one,

$$
\theta^K\theta^L = -\theta^L\theta^K \tag{9.7}
$$

independent of the parameter $s$. The algebra of vector fields then also coincides with the classical one. The form of the matrices

$$
\begin{aligned}
&A = \begin{pmatrix} 1+s & 0 & 0 & s \\ 0 & q & 0 & 0 \\ 0 & 0 & 1/q & 0 \\ -s & 0 & 0 & 1-s \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2s/q & 0 & 0 & 2s/q \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2qs & 0 & 0 & 2qs \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1-s & 0 & 0 & -s \\ 0 & q & 0 & 0 \\ 0 & 0 & 1/q & 0 \\ s & 0 & 0 & 1+s \end{pmatrix}
\end{aligned}
\tag{9.8}
$$

† (8.1) and (8.2) then only make sense in the limit $r \to 1$, $r \neq 1$.

shows that this calculus is nevertheless not trivial. These matrices have some nice properties†

$$BA = BD = B \qquad CA = CD = C \qquad B^2 = C^2 = BC = CB = 0 \tag{9.9}$$

and

$$A^k = A(ks, q^k) \qquad D^l = D(ls, q^l) \qquad AD = DA = \mathrm{diag}(1, q^2, q^{-2}, 1) \tag{9.10}$$

where $A = A(s, q)$ expresses the dependence of the matrix $A$ on the parameters $s$ and $q$. In particular, these properties make it easy to evaluate the action of $\Theta$ on a general monomial which is needed to calculate the co-product (4.45).

In the construction of bicovariant differential calculi on $\mathrm{GL}_{p,q}(2)$ in section 6 we disregarded one case in which we were forced to restrict the parameters $p$ and $q$ by (9.1) (see the paragraph before (6.24)). This additional case is easily worked out and leads to

$$A = D = \begin{pmatrix} 1+s & 0 & 0 & s \\ 0 & q & 0 & 0 \\ 0 & 0 & 1/q & 0 \\ s & 0 & 0 & 1+s \end{pmatrix} \qquad B = C = 0. \tag{9.11}$$

Solving (4.36) we find that also in this case the wedge product and consequently the vector field commutation relations are the classical ones. We know that there is no extension of this calculus to a bicovariant calculus with unconstrained parameters $p$ and $q$. But there may be a left-covariant differential calculus on $\mathrm{GL}_{p,q}(2)$ which for $p = 1/q$ reduces to this bicovariant calculus. The matrix $A$ has the property

$$A^n \equiv (A(s, q))^n = A(s^{(n)}, q^n) \tag{9.12}$$

where

$$s^{(n)} = \tfrac{1}{2}[(1+2s)^n - 1]. \tag{9.13}$$

For $q = 1$ the three examples in this section present non-standard bicovariant differential calculi on the classical Lie group GL(2).

## 10. Conclusions

In order to learn more about non-commutative geometry and in particular differential calculus on quantum groups it is important to elaborate examples. We have tried to treat the two-parameter deformation of GL(2) as systematically and complete as we could. Nevertheless, the reader will certainly find some points where more efforts should have been invested. Already for this rather simple quantum group some calculations could hardly be carried out by hand and we employed the computer algebra system REDUCE [36] to do the job.

† $B^2 = 0$ and $C^2 = 0$ even hold for the general bicovariant calculus.

Our main result is the explicit construction of the most general bicovariant differential calculus on $GL_{p,q}(2)$. It consists of two branches which depend on one new parameter. Different values of this parameter determine different calculi. Even in the classical limit $p, q \to 1$ (which corresponds to the classical group) the new parameter deforms the differential calculus. This parameter also enters the 'Lie algebra' of left-coinvariant vector fields. It remains to be investigated whether this generalized Lie algebra has a Hopf algebra structure and what the uniqueness statement in [13] (see also [37]) concerning quantized Lie algebras has to say about it.

Bicovariance seems to be a very natural condition (see [24]) and it guarantees a consistent higher-order differential calculus. But there may be further or even different conditions to single out differential calculi. Applications of non-commutative differential calculus in physical models should tell us more about this.

One bridge to physical models is due to the expectation that non-local charges in certain conformal and also some massive two-dimensional quantum field theories should be represented by left-coinvariant vector fields on a quantum group, i.e. the generators of the 'quantum Lie algebra'. Also, the geometric approach to BRST symmetry formulated in [40] can be generalized to quantum group symmetries [24] (see also [41]). Here the (quantum) exterior derivative takes the role of the BRST operator and the ghosts are identified with left-coinvariant 1-forms.

Less ambitious than the programme outlined in the introduction is the exploration of Kaluza-Klein type models where extra spacetime dimensions are assumed to form a quantum group. It then has to be worked out how the choice of a differential calculus influences the physical predictions. All the applications of non-commutative differential geometry to elementary particle physics which we are aware of are formulated in a kind of Kaluza-Klein framework [2, 7, 10, 11]. In [10, 11] the extra 'dimensions' just consist of a finite set of points, however.

For the orthogonal quantum groups a matrix representation which defines a bicovariant differential calculus was given in [26] in terms of the $\hat{R}$ matrix†.

$$(\mathscr{F}(M^{\mu}{}_{\nu})^{A}{}_{B}) = (\hat{R}^{\alpha\kappa}{}_{\nu\gamma}(\hat{R}^{-1})^{\beta\mu}{}_{\kappa\delta}). \tag{10.1}$$

This formula gives indeed also a representation of $GL_{p,q}(2)$, but it does not satisfy (4.31). Perhaps similar formulae work.

The quantum group $GL_{p,q}(2)$ is not just a toy model since it contains a deformed $SL(2, \mathbb{C})$. To arrive at a quantum Lorentz group [38, 39] one has to add, however, a generalized complex conjugation, i.e. a $*$-structure [20] on the algebra. Our work still has to be extended taking this additional structure into account (see also [19]).

For a discussion of some technical problems which one meets in the case of non-compact quantum groups we refer to [38].

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† $(\alpha, \beta) \to A$, $(\gamma, \delta) \to B$ is the renumbering of indices which we used in section 3 to express $\hat{R}^{\alpha\beta}{}_{\gamma\delta}$ as a $4\times 4$ matrix.

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